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THE PROBLEM OF DESIGNING LAMINATED PLATES WITH SPECIFIED STIFFNESSES†

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The problem of designing laminated plates, which consists of finding a set of designs that give the plate the stiffness characteristics required is investigated. The problem is considered in a discrete formulation taking account of the constancy of Poisson's ratio and the variability of Young's modulus through the plate thickness. The initial design problem reduces to the problem of convex combinations, for the solution of which the convolution method is employed: © 2000 Elsevier Science Ltd. All rights reserved.

The problem of designing laminated plates has been the subject of a large number of publications. The optimal design problem of obtaining a plate that minimizes any functional (weight, sag, etc.) has been well studied. The problem of finding a method of designing a plate with specified (not necessarily optimal) characteristics [1–3] has been considerably less studied. For a continuous distribution of characteristics through the plate thickness, this problem was examined in [4] using extremum principles.

A solution of the discrete design problem formulated in [2–5] is given below.

1. FORMULATION OF THE PROBLEM

Consider a laminated plate with layers parallel to the coordinate plane and made of homogeneous isotropic materials. It is required to choose the distribution of materials in the layers so as to give the plate specified stiffnesses (stiffnesses in a plane, asymmetrical stiffnesses, and flexural stiffnesses).

The coordinate transverse to the plate will be denoted by y. In the laminated plate Young's modulus $E(y)$ and Poisson's ratio $\nu(y)$ are functions of the variable y.

The *design problem* is as follows. It is required to determine the distribution of the material characteristics that gives the plate a specified stiffness in its plane S^0, asymmetrical stiffness S^1, and flexural stiffness S^2.

For this, it is necessary to solve the problem

$$\int_{-1/2}^{1/2} \frac{E(y)}{1-\nu^2(y)}dy = S^0, \quad \int_{-1/2}^{1/2} \frac{E(y)y}{1-\nu^2(y)}dy = S^1, \quad \int_{-1/2}^{1/2} \frac{E(y)y^2}{1-\nu^2(y)}dy = S^2 \tag{1.1}$$

The integrals in (1.1) give an expression for the appropriate stiffnesses of the plate in terms of the elasticity constants of the layers forming it [2–4].

For simplicity we will assume that $\nu(y) = \text{const}$. Then (1.1) is a problem with regard to $E(y)$. In practice, a finite (often small) number of materials is used.

To discretize the problem, we will stipulate that the plate is divided into m layers of equal thickness $1/m$, and the function $E(y)$ is constant in the ranges $[-1/2 + (i-1)/m, -1/2 + i/m)$. Here and below, $i = 1, \ldots, m$.

The required quantities are E_i. In the physical sense, $E_i \geq 0$. Problem (1.1) can be reduced to the following

$$\begin{gathered}\sum_i x_i = 1, \; x_i = E_i / mS^0 > 0, \; \sum_i \mathbf{v}_i x_i = \mathbf{v} \\ \upsilon_{il} = \frac{1}{m}\int_{-1/2+(i-1)/m}^{1/2+i/m} y^l dy, \; \upsilon_l = S^l / S^0, \; l = 1,2 \; (\mathbf{v}_i = (\upsilon_{i1}, \upsilon_{i2}), \; \mathbf{v} = (\upsilon_1, \upsilon_2))\end{gathered} \tag{1.2}$$

2. THE DISCRETE PROBLEM OF CONVEX COMBINATIONS

We will consider the following problem. Let $Z_n \subset [0, 1]$ be a finite set (consisting of n numbers), and let $\mathbf{v}_i$ and $\mathbf{v} \in R^k$ be specified vectors. It is required to determine the numbers x_i that are the solution of the following problem

$$\sum_i \mathbf{v}_i x_i = \mathbf{v} \tag{2.1}$$

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$$\sum_i x_i = 1 \tag{2.2}$$

$$x_i \in Z_n, \quad i = 1, \ldots, m \tag{2.3}$$

Problem (2.1), (2.2) with the condition

$$0 \leq x_i \leq 1 \tag{2.4}$$

is the problem of convex combinations (PCC) examined earlier [1, 2]. Problem (2.1)–(2.3) is a discrete PCC.

The purpose of the present paper is to construct a general solution of problem (2.1)–(2.3), i.e. the set of all coefficients $x_i \in Z_n$ of the convex combinations of the vectors $\mathbf{v}_i$ that give the vector $\mathbf{v}$.

We will remove the discreteness condition (2.3) and replace it with condition (2.4). The general solution of the PCC (2.1), (2.2), (2.4) is well known and has the form [1, 2]

$$x_i = \sum_{\gamma=1}^{M} P_{i\gamma}\lambda_\gamma, \quad i = 1, \ldots, m \tag{2.5}$$

where $\mathbf{P}_\gamma = (P_{1\gamma}, \ldots, P_{m\gamma})$ is a certain finite set of solutions of the PCC (2.1), (2.2), (2.4), and λ_γ ($\gamma = 1, \ldots, M$) are any numbers satisfying the conditions

$$\sum_{\gamma=1}^{M} \lambda_\gamma = 1, \quad 0 \leq \lambda_\gamma \leq 1 \tag{2.6}$$

In order words, the set $\Lambda(\mathbf{v})$ of solutions of the PCC(2.1),(2.2),(2.4) has the form

$$\Lambda(\mathbf{v}) = \text{conv}\,\{\mathbf{P}_\gamma, \gamma = 1, \ldots, M\}.$$

A method for constructing the system of vectors $\{\mathbf{P}_\gamma, \gamma = 1, \ldots, M\}$ was described earlier [1, 2], and examples of its use are given in [3–5]. It was shown [2] that, for random perturbations of the data in the PCC (2.1),(2.2),(2.4) with a probability of unity, the system $\{\mathbf{P}_\gamma, \gamma = 1, \ldots, M\}$ is identical with the set of extreme points of the polyhedron $\Lambda(\mathbf{v})$ of the solutions of the PCC (2.1),(2.2), (2.4). This means that $\{\mathbf{P}_\gamma, \gamma = 1, \ldots, M\}$ is the minimum system of points producing a set of solutions of the PCC (2.1),(2.2),(2.4).

The set $Z_n^m = \{\mathbf{x}: x_i \in Z_n\}$ is a discrete grid in R^m. The set of solutions of the PCC (2.1)–(2.3) is $\Lambda(\mathbf{v}) \cap Z_n^m$.

Problem (2.1)–(2.3) will be solved if we indicate the vectors given by formulae (2.5) and (2.6) and which satisfy the condition $x_i \in Z_n$.

Relations (2.5) and (2.6) can be regarded as a PCC with respect to λ_γ. The convolution algorithm proposed earlier [1, 2] indicates the following [which also follows from the convexity of the set $\Lambda(\mathbf{v})$]: if the first $i - 1$ equations in (2.5) are satisfied, then the following ith equation is solvable when, and only when,

$$x_i \in I_i = [\min_i, \max_i] \tag{2.7}$$

Generally speaking, the interval I_i depends on $x_1, \ldots, x_{i-1}$.

From (2.7) we obtain the necessary adn sufficient condition for the existence of a solution of the discrete PCC:

$$Z(i) = Z_n \cap I_i \neq \varnothing \quad \text{for all } i \tag{2.8}$$

Since the intervals I_i depend on $x_1, \ldots, x_{i-1}$, a tree T arises. Its apex $T(0)$ corresponds to $i = 1$, the absence of a solution. Branching on the level $T(i - 1)$ corresponds to the points $Z(i)$. Any branch going from the root $T(0)$ to the level $T(m)$ gives the solution of the discrete PCC (2.1)–(2.3). On the other hand, the branch from the root $T(0)$ to the level $T(m)$ corresponds to any solution of the discrete PCC. Thus, the given tree gives a set of all solutions of the discrete PCC.

3. NUMERICAL ALGORITHMS

Changing from the PCC (2.1),(2.2),(2.4) *to the PCC* (2.5),(2.6) – *construction of the system of vectors* $\{\mathbf{P}_\gamma\, \gamma = 1, \ldots, M\}$. The system of vectors can be constructed using the convolution algorithm described earlier in [1, 2]. Its properties (in particular, the property of optimality) have been described above. More details can be found in [1–5].

Checking the solvability of the PCC (2.5),(2.6) *for the right-hand sides of the set* Z_n – *construction of the* *intervals* I_i. The simplex method can be used to construct the intervals [mix$_i$, max$_i$. For this, at the $(i-1)$th step, the first $i-1$ equations from (2.5) and Eq. (2.6) must be regarded as constraints, the objective function $L(\lambda)$ must be constructed on the basis of the ith equation from (2.5)

$$L(\boldsymbol{\lambda}) = \sum_{\gamma=1}^{M} P_{i\gamma}\lambda_\gamma \tag{3.1}$$

and the problem

$$L(\boldsymbol{\lambda}) \to \min(\max) \tag{3.2}$$

must be considered.

The solution of problem (3.2)(maximization and mimimization) by the simplex method requires less computer memory than the convolution method.

Construction of a tree. A tree can be constructed by any method, since, in practice, its size proves not to be very large (see below).

Thus, the numerical algorithm for solving the discrete PCC has been reduced to solving well-known problems – a PCC and a problem of linear programming.

4. THE SOLVABILITY OF THE PCC AND THE STRUCTURE OF ITS SOLUTION

Consider an internal point [6] $\mathbf{x}$ of the set $\Lambda(\mathbf{v})$, i.e. a point for which the vector $\mathbf{y}$ and the number δ are such that $\mathbf{x} + \mathbf{y}\tau$ lies in $\Lambda(\mathbf{v})$ when $0 < \tau < \delta$. Substituting $\mathbf{x} + \mathbf{y}\tau$ into relations (2.1) and (2.2) and differentiating with respect to τ, we obtain that the vector $\mathbf{y}$ satisfies the equalities

$$\sum_i \mathbf{v}_i y_i = 0, \quad \sum_i y_i = 0 \tag{4.1}$$

Introducing the vectors $\mathbf{w}_i = (v_{l1}, \ldots, v_{lm})$ and $\mathbf{w}_0 = (1, \ldots, 1)$, we can rewrite (4.1) in the form $\mathbf{w}_l\mathbf{y} = 0$ $(l = 0, 1, 2)$ for each vector $\mathbf{y}$ connecting l internal points. This means that the set $\Lambda(\mathbf{v})$ lies in the hyperplane specified by Eqs (4.1) and has a dimensionality less than m (remember that the solutions of PCC (2.1),(2.2),(2.4) are the elements R^m).

By virtue of this property problem (2.5),(2.6) for an arbitrarily specified x will generally not have solutions since the random incidence of the point on the hyperplane has a probability of zero.

In view of this, it is possible to propose a perturbation of the set $\{\mathbf{P}_\gamma\ \gamma = 1, \ldots, M\}$ with the aim of giving the convex shell of the perturbed set solidity (a dimensionality of m). We conclude from (4.1) that the set $\{\mathbf{P}_\gamma\ \gamma = 1, \ldots, M\}$ should be perturbed by the vectors $\mathbf{w}_i = (\mathbf{v}_i, 1)$ [which are solutions of (4.1) and "perpendicular" to $\Lambda(\mathbf{v})$].

We will consider a perturbation of the set $\{\mathbf{P}_\gamma\ \gamma = 1, \ldots, M\}$ of the form $\{\mathbf{P}_\gamma + r\xi\mathbf{w}_{\gamma \bmod 3}$, $\gamma = 1, \ldots, M\}$, where ξ is a random quantity uniformly distributed in the segment [0, 1], r is the characteristic magnitude of the perturbation and mod denotes division with respect to modulus. With a probability of unity, conv $\{\mathbf{P}_\gamma + r\xi\mathbf{w}_{\gamma \bmod 3}\}$ has a dimensionality of m.

Suppose problem (2.5) with the vectors $\{\mathbf{P}_\gamma + r\xi\mathbf{w}_{\gamma \bmod 3}, \gamma = 1, \ldots, M\}$ instead of $\{\mathbf{P}_\gamma\ \gamma = 1, \ldots, M\}$ has the solution $\mathbf{x}$. This solution can be represented in the form of the convex combination

$$\mathbf{x} = \sum_{\gamma=1}^{M} (\mathbf{P}_\gamma + r\xi\mathbf{w}_{\gamma \bmod 3})\lambda_\gamma \tag{4.2}$$

Substituting this expression into (2.1) and taking account of the fact that $\{\mathbf{P}_\gamma\ \gamma = 1, \ldots, M\}$ satisfy equalities (2.1), we obtain

$$\left|\sum_{\gamma=1}^{M} \sum_i \mathbf{v}_i(\mathbf{P}_{i\gamma} + r\xi\mathbf{w}_{i\gamma \bmod 3}) - \mathbf{v}\right| \leqslant \sum_{\gamma=1}^{M} \sum_i \left|\mathbf{v}_i r\xi w_{i\gamma}\right|$$

This expression is the error with which the solution $\mathbf{x}$ satisfies Eq. (2.1). It can be seen that it is of order r.

5. NUMERICAL TESTS

Programs were written to obtain simplicial solutions of PCC, to construct a tree, and to solve the problem of linear programming.

Test problem. The following was used as the test problem. A certain structure of the laminated plate, E_i^*, was specified and the stiffnesses for it, S^ν $(\nu = 0, 1, 2)$, were estimated by means of formulae (1.1). Then, the design problem was solved for these stiffnesses. A solution for E_i^* should exist in the set of solutions of the design problem.

This property was also checked. Solutions were obtained for a number of layers $m = 8$–12. The number of solutions, M, of the PCC (2.1),(2.2),(2.4) did not exceed 100. As a consequence, the PCC (2.5),(2.6) was in the form of 8–12 equations with a maximum number of variables of 100. In the examples checked, the solution were unique; E_i^*. This corresponds to what was stated in Section 4.

A problem with a perturbed system $\{\mathbf{P}_\gamma\ \gamma = 1, \ldots, M\}$. We will give the following problem as a typical example. The number of layers $m = 7$. The design $\{E_i^*, i = 1, \ldots, 7\} = \{3, 5, 3, 5, 3, 5, 3\}$ will be assumed to be known. The corresponding stiffnesses $S^0 = 3.8571$, $S^1 = 0$, and $S_2 = 1.1348$. The number of solutions of the PCC (2.1),(2.2),(2.4) is $M = 12$.

The set of materials $Z_6 = \{1, 2, 3, 5, 7, 10\}$ is used for the design.

The perturbation parameter was taken to be equal to $r = 0.05$. The number of nodes of the tree T on the various levels is as follows:

Level i	0	1	2	3	4	5	6	7
Number of nodes	1	5	19	49	157	80	12	4

The following designs were obtained: $E_1 = \{5, 3, 5, 5, 1, 7, 1\}$, $E_2 = \{3, 5, 5, 5, 1, 5, 3\}$, $E_3 = \{3, 5, 3, 5, 3, 5, 3\}$, and $E_4 = \{1, 7, 3, 5, 3, 3, 5\}$. The design E_3 matches the initial design.

The stiffnesses corresponding to the designs obtained are $S^0 = 3.857$ and $S^2 = 1.135$ for all the designs, while S^1 is equal to 0.163, 0.245 and 0.082.

6. OTHER DESIGN PROBLEMS

A problem with stiffnesses from a specified interval. As noted in Section 4, there is often no solution to the design problem. In numerical calculations it can be seen that, in a number of cases, solutions arise that give plate stiffnesses close to those required but not precisely equal to them [7]. We will weaken the equalities by requiring that the stiffnesses S^ν ($\nu = 0, 1, 2$) belong to specified intervals $[S^\nu - \delta S^\nu, S^\nu + \delta S^\nu]$ ($\nu = 0, 1, 2$). After the additional variables $x_{m+1}, \ldots, x_{m+5}$ have been introduced, the inequalities

$$S^\nu - \delta S^\nu \leqslant \sum_i d_{\nu i} x_i \leqslant S^\nu + \delta S^\nu$$

reduce to the following PCC

$$\begin{aligned}
&\sum_i x_i = 1, \ x_i \geqslant 0 \\
&\sum_i v_{li} x_i - x_{m+2l-1} = v_l \\
&\sum_i v_{li} x_i + x_{m+2l} = v_l + \delta S^l/(S^l - \delta S^l)
\end{aligned} \tag{6.1}$$

The quantities $\{v_{li}\}$ are defined above, and

$$x_i = (E_i/m)(S^0 - \delta S^0), \ v_l = (S^l + \delta S^l)/(S^l - \delta S^l), \ l = 1, 2$$

The problem of the discrete optimization pf plates. Let the objective function (weight or price) $M(\mathbf{x})$ be specified for the plate. It is required to solve the problem of minimizing $M(\mathbf{x})$ with constraints (2.1)–(2.3). To do this, we construct a set X of solutions of the PCC (2.1)–(2.3) by the method set out above, after which the solution is obtained by minimizing the function $M(\mathbf{x})$ in the finite set X.

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